

Noncommutative deformations and perverse sheaves

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- The basic idea is to understand structures on categories that appear on LG mirrors in terms of categories of coherent sheaves.
- Perverse schober are a fundamental example of this sort of structure.
- We will show that perverse schobers show up in algebraic geometry whenever one has a semiorthogonal decomposition of a triangulated category with Serre functor.
- The expression of semiorthogonal decompositions in terms of perverse schobers suggests an easy way to understand noncommutative deformations of rational surfaces.
- Attempts to generalize this construction lead to sheaves of categories on higher-dimensional complexes which also have an interpretation in terms of mirror symmetry.

- Perverse sheaves are objects in the bounded derived category of constructible sheaves on a complex manifold whose cohomology obeys certain support conditions.
- In many cases, they can be defined in terms of combinatorial data.
- For instance, the standard example is of perverse sheaves on the complex disc which are local systems outside of the point 0. Such objects are in bijection with the data:
 - Pairs of vector spaces V and W
 - Pairs of maps $f : V \rightarrow W$ and $g : W \rightarrow V$satisfying the condition that $\text{id}_V - g \cdot f$ is invertible.

- Galligo, Granger and Maisonobe (1985) show more. They take the skeleton given by the positive real line in the disc with stratification

$$S_0 = \{0\}, \quad S_1 = \{z \in D : \operatorname{Im}(z) = 0, \operatorname{Re}(z) > 0\}$$

and show that there's a functor from the category of perverse sheaves on D constructible with respect to the stratification T to the category of sheaves on K constructible with respect to the stratification S given by the data described on the previous slide.

- Kapranov and Schectman (2014) approach the question of how one should think about perverse sheaves with coefficients in categories.
- Their idea is roughly that it is hard to define a perverse sheaf of categories directly, but defining a constructible sheaf of categories (especially on a simple cell complex) is not so difficult to describe.

Definition:

A perverse schober on the disc with stratification $(D, 0)$ is just the data of a pair of pretriangulated dg categories \mathbf{A} and \mathbf{B} and F spherical functor from \mathbf{A} to \mathbf{B} .

Definition:

A functor F of pretriangulated dg categories is spherical if it has right and left adjoints R and L and

- 1 The functor $T = \text{cone}(FR \rightarrow \text{id}_{\mathbf{B}})$ is an autoequivalence,
- 2 The natural transformation $LT[-1] \rightarrow LFR \rightarrow R$ is an equivalence of functors

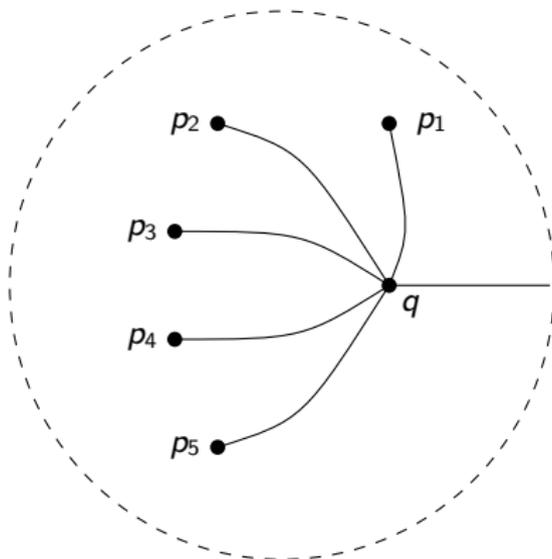
More generally,

Definition:

A (K -coordinatized) perverse schober on the disc with respect to the stratification $(D, \{p_1, \dots, p_k\})$ is given by an ordered set of categories \mathbf{A}_i , a category \mathbf{B} and spherical functors $F_i : \mathbf{A}_i \rightarrow \mathbf{B}$ along with a set of nonintersecting paths γ_i from a basepoint $q \in D$ to each p_i and a path γ_∞ from q to the boundary of D .

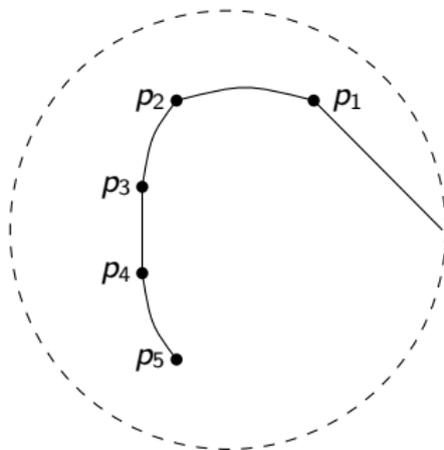
- There's a braid group action on the set of such paths and a corresponding braid group action on the set of all perverse schobers.
- This action says that we can exchange the order of \mathbf{A}_i and \mathbf{A}_{i+1} at the price of changing F_i to $T'_{i+1}F_i$.

The data of a perverse schober gives rise to a constructible sheaf of categories on the following skeleton



The fiber at p_i is the category \mathbf{A}_i , at every point away from p_i and q is just \mathbf{B} . The fiber over the point q is a category that I'll call $S_n(\mathbf{B})$. This is some sort of version of representations of the A_n quiver in \mathbf{B} .

We will, however, use a different skeleton and constructible sheaf of categories for convenience in the future,



- Let us define $\langle \mathbf{A}_i, \mathbf{B} \rangle_{F_i}$ to be the category of triples (a, b, μ) where $a \in \mathbf{A}_i, b \in \mathbf{B}$ and $\mu \in Z^0 \mathbf{B}(F(a), c)$.
- This constructible sheaf of categories here assigns to each point $p_i, (i \neq 5)$ the category $\langle \mathbf{A}_i, \mathbf{B} \rangle$.
- To p_5 we assign the category \mathbf{A}_5 and to each point in the edges we assign the category \mathbf{b} .

Semiorthogonal decompositions

Perverse schobers can be thought of as a way of encoding semiorthogonal decompositions.

Definition:

A collection of admissible subcategories $\mathbf{A}_1, \dots, \mathbf{A}_n$ of \mathbf{A} forms a semiorthogonal decomposition of \mathbf{A} if for every object a in \mathbf{A} there is a collection of triangles

$$a_1 \xrightarrow{\mu_1} a_2 \rightarrow b_1, \text{ and } a_{i+1} \xrightarrow{\mu_i} b_{i-1} \rightarrow b_i$$

with $a_i \in \mathbf{A}_i$ and $b_n = a$ and $\mathbf{A}(a_j, a_i) = 0$ if $j > i$.

The category of global sections of the constructible sheaf of categories associated to any perverse schober admits a semiorthogonal decomposition into $\mathbf{A}_1, \dots, \mathbf{A}_n$.
A theorem of Addington says that

Theorem

if $F : \mathbf{A} \rightarrow \mathbf{C}$ is a spherical functor with cotwist the Serre functor up to shift, then if \mathbf{A} admits a semiorthogonal decomposition into subcategories $j_i : \mathbf{A}_i \rightarrow \mathbf{A}$, then the functors $F \cdot j_i$ are spherical with cotwist the Serre functor on \mathbf{A}_i up to shift.

Therefore, for any category with semiorthogonal decomposition and functor F as above, one can construct a perverse schober from this data.

- **Example:** Y is the vanishing locus of a section of the anticanonical bundle of a variety X . Then if $i : Y \rightarrow X$ is the embedding, then $i^* : D^b(\text{coh } X) \rightarrow D^b(\text{coh } Y)$ is spherical with cotwist S_X up to shift.
- If X is Fano then there is a semiorthogonal decomposition of $D^b(\text{coh } X)$ so such a decomposition appears naturally.
- E. Segal proved recently that for any autoequivalence of a category, one can construct a spherical functor whose twist functor is that autoequivalence.
- Using this, if \mathbf{A} admits a Serre functor, we can always find a category \mathbf{C} and a functor $F : \mathbf{A} \rightarrow \mathbf{C}$ whose cotwist is a shift of the Serre functor. The following is then not so difficult to prove.

Theorem

Any triangulated dg category \mathbf{A} with a semiorthogonal decomposition \mathbf{A}_\bullet and a spherical functor $\mathbf{A} \rightarrow \mathbf{C}$ whose cotwist is the Serre functor on \mathbf{A} up to twist is the category of global sections of a perverse schober on D, p_1, \dots, p_k with points corresponding to components of the SOD.

- Here, global sections are obtained by taking the homotopy limit of the functors and categories involved in our constructible sheaf.

- **Example:** If we take \mathbb{P}^2 and an smooth cubic curve E in \mathbb{P}^2 , then the functor $i^* : D^b(\text{coh } \mathbb{P}^2) \rightarrow D^b(\text{coh } E)$ is spherical with cotwist the Serre functor on \mathbb{P}^2 up to shift.
- The Beilinson exceptional collection, $\mathcal{O}, \mathcal{O}(1)$ and $\mathcal{O}(2)$ on \mathbb{P}^2 gives rise to a semiorthogonal decomposition on $D^b(\text{coh } \mathbb{P}^2)$, hence spherical functors $F_i : D^b(k) \rightarrow D^b(\text{coh } E)$ for $i = 1, 2, 3$.
- According to the theorem in the previous section, we can recover $D^b(\text{coh } \mathbb{P}^2)$ from F_1, F_2 and F_3 as the category of global sections of the corresponding perverse schober.

- The useful thing about elliptic curves is that a spherical object has $\dim \text{Ext}^1(S, S) = 1$.
- Therefore, it is possible to deform the functors F_i . These deformations are obtained by composing with an automorphism $\tau \in \text{Pic}^0(E)$ of E .
- Formally: for any quadruple of the form (E, \mathcal{L}, τ) for E a (smooth) cubic curve, \mathcal{L} a degree 3 ample line bundle on E and $\tau \in \text{Pic}^0(E)$, we can build a perverse schober given by the spherical functors G_1, G_2 and G_3 corresponding to the spherical objects in $D^b(\text{coh } E)$ given by \mathcal{O}, \mathcal{L} and $\tau^* \mathcal{L}^2$.
- This builds the category of coherent sheaves on noncommutative \mathbb{P}^2 .
- A closely related construction was described by Bondal and Polishchuk (1993) in terms of elliptic exceptional collection. They showed that from this one obtains all noncommutative deformations of \mathbb{P}^2 .
- The same procedure can be used to produce noncommutative deformations of any rational surface with a smooth anticanonical section.
- However, we cannot use the same method to produce deformations of higher dimensional projective spaces since spherical objects on higher dimensional Calabi-Yau varieties do not admit infinitesimal deformations.

- In order to build a noncommutative deformation of \mathbb{P}^3 (say) we can build a noncommutative deformation of an anticanonical divisor in \mathbb{P}^3 which deforms with the spherical functors coming from restriction of admissible subcategories of $D^b(\text{coh } \mathbb{P}^3)$.
- This gives a perverse schober whose category of global sections should be a deformation of $D^b(\text{coh } \mathbb{P}^3)$.
- In general, noncommutative projective spaces do not have smooth anticanonical divisors, according to Kanazawa, so this should not work with noncommutative K3s/CY3s etc.
- Therefore, in order to produce noncommutative deformation of \mathbb{P}^3 using perverse schobers, we should take either a union of a pair of quadrics in \mathbb{P}^3 or a hyperplane and a cubic, then try to produce noncommutative deformations of these varieties which deform with specific spherical functors.

- The functor $i^* : D^b(\text{coh } \mathbb{P}^3) \rightarrow \text{Perf}(Q_1 \cup Q_2)$ is spherical with cotwist equal to the Serre functor on \mathbb{P}^3 up to twist.
- Therefore the reconstruction theorem from the previous section can be applied to recover $D^b(\text{coh } \mathbb{P}^3)$.
- We may build noncommutative deformations of quadric hypersurfaces in \mathbb{P}^3 in much the same way we build noncommutative deformations of \mathbb{P}^2 if we can build noncommutative deformations of $\text{Perf}(Q_1 \cup Q_2)$ which deform these spherical objects.
- Smooth quadric hypersurfaces in \mathbb{P}^3 are just $\mathbb{P}^1 \times \mathbb{P}^1$, so they admit a semiorthogonal decomposition

$$\langle \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1) \rangle$$

- These bundles give spherical functors F_1, \dots, F_4 from $D^b(k)$ to $D^b(\text{coh } E)$ so that the image of the complex k is \mathcal{O}_E , two degree 2 line bundles \mathcal{L}_1 and \mathcal{L}_2 and $\mathcal{L}_1 \otimes \mathcal{L}_2$ respectively.
- The corresponding perverse schober has category of global sections equal to $D^b(\text{coh } Q)$.

- The category $\text{Perf}(Q_1 \cup Q_2)$ is a homotopy fiber product of $D^b(\text{coh } Q_1)$ and $D^b(\text{coh } Q_2)$ over $D^b(\text{coh } E)$ where E is the elliptic curve in their intersection.
- The spherical objects in $\text{Perf}(Q_1 \cup Q_2)$ can be built out of exceptional objects on Q_1 and Q_2 .
- It's well known that if one has an exceptional object E on a category \mathbf{A} and spherical functor $F : \mathbf{A} \rightarrow \mathbf{B}$ whose cotwist is the Serre functor (up to shift) then $F(E)$ is spherical on \mathbf{B} (this goes back to Seidel-Thomas).

Theorem

If $F_1 : \mathbf{A}_1 \rightarrow \mathbf{B}$ and $F_2 : \mathbf{A}_2 \rightarrow \mathbf{B}$ are spherical functors whose cotwists $S_1[k]$ and $S_2[k]$ respectively, and if E_1 and E_2 are exceptional objects on \mathbf{A}_1 and \mathbf{A}_2 respectively with $F_1(E_1) \cong F_2(E_2)$, then E_1 and E_2 induce a spherical object in the homotopy fiber product $\mathbf{A}_1 \times_{\mathbf{B}}^h \mathbf{A}_2$.

- In particular, if we have a category \mathbf{A} with an exceptional object E on it and a spherical functor $F : \mathbf{A} \rightarrow \mathbf{B}$, then there is a spherical object S on $\mathbf{A} \times_{\mathbf{B}}^h \mathbf{A}$ corresponding to E .

- Noncommutative deformations (denoted $D^b(\text{coh } Q_\mu)$) of the quadric $D^b(\text{coh } Q)$ can be obtained by taking the spherical functors F_1, \dots, F_4 and deforming F_4 by composing with an automorphism $\tau \in \text{Pic}^0(E)$.
- This gives a family of perverse schobers on the complex



- Call the categories of global sections of such schobers $D^b(\text{coh } Q_\mu)$.
- There is always a functor $D^b(\text{coh } Q_\mu) \rightarrow D^b(\text{coh } E)$ with cotwists the Serre functor (restriction to the fiber on the left hand edge).
- Therefore, any exceptional objects $D^b(\text{coh } Q_\mu)$ must give rise to exceptional objects on the category $D^b(\text{coh } Q_\mu) \times_{D^b(\text{coh } E)}^h D^b(\text{coh } Q_\mu)$.
- In particular we have exceptional objects in $D^b(\text{coh } Q_\mu)$ coming from spherical objects $\mathcal{O}, \mathcal{L}_1, \mathcal{L}_2$ and $\tau^*(\mathcal{L}_1 \otimes \mathcal{L}_2)$. These extend to spherical objects on the homotopy fiber product.

- In the case where Q_μ is commutative, the spherical objects in $\text{Perf}(Q_1 \cup Q_2)$ which recover $D^b(\text{coh } \mathbb{P}^3)$ are the restrictions of $\mathcal{O}(i)$ to $Q_1 \cup Q_2$ for $i = 0, 1, 2, 3$.
- These spherical objects can also be described in terms of exceptional objects in $D^b(\text{coh } Q_1)$ and $D^b(\text{coh } Q_2)$. These are the spherical objects coming from $\mathcal{O}_{Q_j}(i, i)$ for $i = 0, 1, 2, 3$.
- As Q deforms noncommutatively, these spherical objects deform as well. We have exceptional objects $F_1(k)$, $F_4(k)$, $S(F_1(k))$ and $S(F_4(k))$ (for S the Serre functor on $D^b(\text{coh } Q_\mu)$) which give spherical objects in the homotopy fiber product.

Theorem

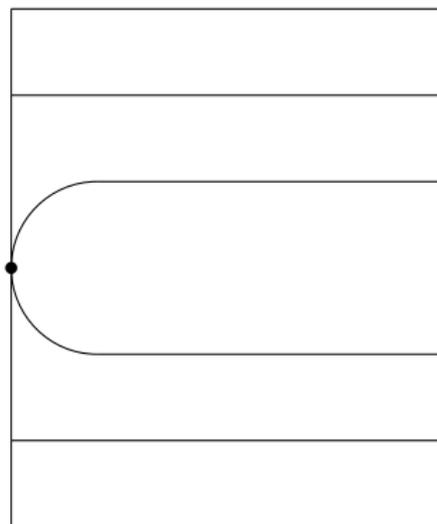
For any noncommutative deformation \mathbf{C} of the quadric hypersurface there is a perverse schober on the disc with respect to the stratification (D, p_1, p_2, p_3, p_4) and generic fiber \mathbf{C} whose category of global sections recovers a noncommutative deformation of \mathbb{P}^3 .

- Therefore, noncommutative deformations of \mathbb{P}^3 can also be obtained from deformations of an underlying perverse schober.
- In the case of \mathbb{P}^2 we deformed the spherical functors and kept the fiber constant, but in this case we deform the fiber in a way that deforms with the spherical functors.
- Fibers of this perverse schober can be written as global sections of a constructible sheaf of categories on the complex



- So these perverse schobers have generic fibers which themselves are global sections of a sheaf of categories on 1-dimensional complex. Can we extend this so that there is a 2-dimensional cell complex which encodes all of this data at the same time.

There's an easy way to encode certain spherical objects in the fiber products displayed above. We can build the following two-dimensional complex



- Here, along the horizontal edges, we assign the categories $\langle \mathbf{A}_i, \mathbf{C} \rangle_{F_i}$.
- To the vertical edges, we assign the trivial category.
- To the point, we assign the category \mathbf{A}_1 .
- This construction only works if the two categories on the central edges coming together are the same and the functors are the same.
- So in the case that we mentioned earlier, we can encode the spherical object \mathcal{O} on $\text{Perf}(Q_1 \cup Q_2)$ in terms of a two-dimensional complex directly.
- There are many semiorthogonal decompositions of a given category, once we have one, each obtained by mutation from the other.
- Therefore, there's lots of different perverse schobers encoding the same category.

- We want to give an interpretation of these mutations in terms of constructible sheaves of categories on a 2-dimensional complex.

Theorem

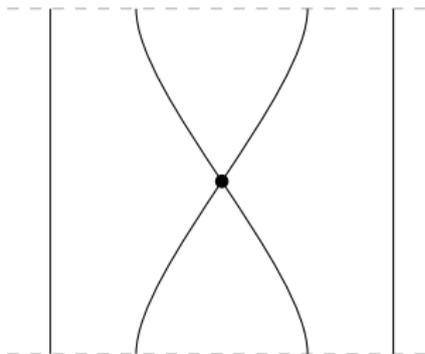
Given a perverse schober given by the data of spherical functors \mathbf{A}_i, F_i , the category of global sections of this perverse schober is the same as that of the perverse schober specified by the data $\tilde{\mathbf{A}}_i, \tilde{F}_i$ where

$$\tilde{\mathbf{A}}_i = \mathbf{A}_i \text{ if } i \neq j, j+1 \text{ and } \tilde{\mathbf{A}}_j = \mathbf{A}_{j+1}, \tilde{\mathbf{A}}_{j+1} = \mathbf{A}_j, \tilde{F}_j = T_{j+1}F_{j+1}, \tilde{F}_{j+1} = F_j.$$

where T_i denotes the spherical twist associated to F_i .

- This is equivalent to saying that Kapranov and Schectman's equivalence of perverse schobers under braid group action does not affect the category of global sections.
- This equivalence can be encoded by a two-dimensional cell complex equipped with a constructible sheaf of categories.

- This can be described using a sheaf of categories on the 2-dimensional cell complex



- (the sheaf of categories determines whether this encodes a right or left mutation).
- So we can use this mutation complex along with the complex encoding spherical objects to build a 2-dimensional cell complex with constructible sheaf of categories which encodes each of the spherical functors in the construction of our noncommutative deformation of \mathbb{P}^3 .
- This sheaf of categories has category of global sections $D^b(\text{coh } \mathbb{P}^3)$.
- Deforming the spherical functors to $D^b(\text{coh } E)$ induces a noncommutative deformation of $D^b(\text{coh } \mathbb{P}^3)$.

- There's a standard LG model mirror of \mathbb{P}^3 given by the pair of $(\mathbb{C}^\times)^3$ and the Laurent polynomial

$$f(x, y, z) = x + y + z + \frac{1}{xyz}$$

- Each monomial in this expression corresponds to a toric boundary divisor in the mirror.
- The sum of the toric boundary divisors in $\text{Pic}(\mathbb{P}^3)$ is equal to the class of the anticanonical divisor on \mathbb{P}^3 .
- We are interested in the pair $\mathbb{P}^3, Q_1 \cup Q_2$, so we are naturally led to decomposing $f(x, y, z)$ into a pair of functions

$$f_1(x, y, z) = x + y, \quad f_2(x, y, z) = z + \frac{1}{xyz}.$$

- This gives us a map $(f_1, f_2) : (\mathbb{C}^\times)^3 \rightarrow \mathbb{C}^2$ whose fibers are punctured elliptic curves.
- There's a map g from \mathbb{C}^2 to \mathbb{C} sending (t_1, t_2) in \mathbb{C}^2 to $t_1 + t_2$. The composition $g \circ (f_1, f_2)$ recovers f .
- We may compactify $(\mathbb{C}^\times)^3$ relative to this map to obtain an elliptic fibration over \mathbb{C}^2 .
- This fibration is degenerate along the curve

$$\Delta = \{t_1 t_2 (t_1^2 t_2^2 - 16) = 0\}$$

- At the points $t_1 = 0, t_2 \neq 0$ and $t_2 = 0, t_1 \neq 0$, we have a chain of two rational curves, $t_1 = t_2 = 0$, we have a chain of four rational curves and elsewhere there's just nodal elliptic curves.
- The map g sends Δ to \mathbb{C} and is ramified over the point 0 and the points $p_i = ce^{ni\pi/2}$ for $n = 0, 1, 2, 3$ and some constant c .

- The topology of the pair (\mathbb{C}, Δ) is determined by certain braid group actions on the fibers of Δ under the map g to \mathbb{C} .
- Take a base point q in \mathbb{C} and take a set of paths γ_i from q to the points $0, p_0, p_1, p_2, p_3$ where $g|_{\Delta}$ ramifies.
- Mark the six points in Δ above q and trace them along γ_i .
- Along each path γ_i , the six points braid, and a pair of p_i 's collide.
- This braiding and collision is enough to determine the topology of the pair (\mathbb{C}^2, Δ) .
- In the case of \mathbb{P}^3 , this data is somewhat complicated, but reduces to *precisely the information of the mutations recorded above* (if we ignore the point 0).

- The idea is then that the fibration (f_1, f_2) provides a “perverse sheaf of Fukaya categories” on \mathbb{C}^2 which is a local system outside of Δ .
- One can build a combinatorial skeleton for the pair (\mathbb{C}, Δ) out of the braiding data.
- The data of this perverse sheaf of Fukaya categories restricts to the data of a constructible sheaf of categories on the skeleton and this constructible sheaf of categories is what is being encoded by the skeleton on the other side.
- The category of global sections of this perverse sheaf of categories then should recover the Fukaya-Seidel category of the fibration.
- This has to do with the fact that we use the map f implicitly in the construction of this skeleton.